Lattices, Shellings and Matroids- Oh My!

Olivetti by Marisa Belk

Now in Technicolor!!!

Simplicial Complexes

A **simplicial complex** on a set V (vertices) is a set Δ of subsets of V (faces) such that:

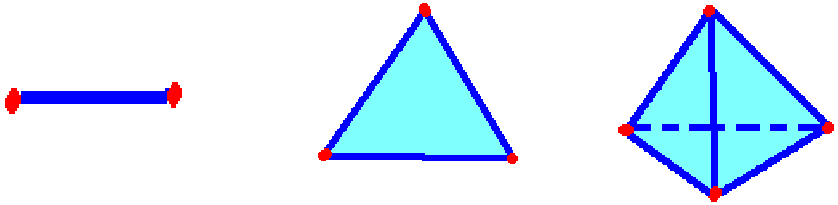
1) If $v \in V$, then $\{v\} \in \Delta$
2) If $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$

The **dimension** of a face F is $|F| - 1$

A **facet** is a maximal face of Δ

A complex is **pure** if every facet has the same dimension

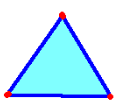
Or take simplices, and glue them together along faces



The f-vector of a Simplicial Complex

It is convenient to keep track of the number of faces in a simplicial complex using f_i = number of faces of cardinality i. (f_0 = 1)

A useful bookkeeping device can be the generating function:
$f_\Delta(x) = x^r + f_1 x^{r-1} + f_2 x^{r-2} \ldots + f_r$ where $|V| = r$

$f_\Delta($  $) = x^3 + 3x^2 + 3x + 1$

The **Reduced Euler Characteristic**
$\chi(\Delta) = (-1)^{r-1} f_\Delta(-1)$, since this is the alternating sum of the number of faces in a given dimension minus one.

Note: The Reduced Euler Characteristic is the alternating sum of betti numbers (homology dimension) of a space minus 1

Partially Ordered Sets

A **Poset** P is a set together with a binary operation $\leq$ that is:

1) Reflexive: $x \leq x$

2) Transitive: If $x \leq y$ and $y \leq z$, then $x \leq z$

3) Antisymmetric: If $x \leq y$ and $y \leq x$, then $x = y$

y **covers** x if $x < y$ and there is no $z \in P$ such that $x < z < y$

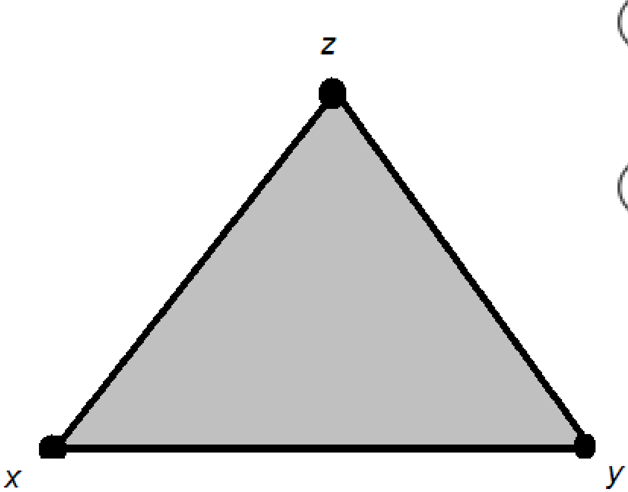
Two elements x, y are **comparable** if $x \leq y$ or $y \leq x$.
Otherwise, they are **incomparable**

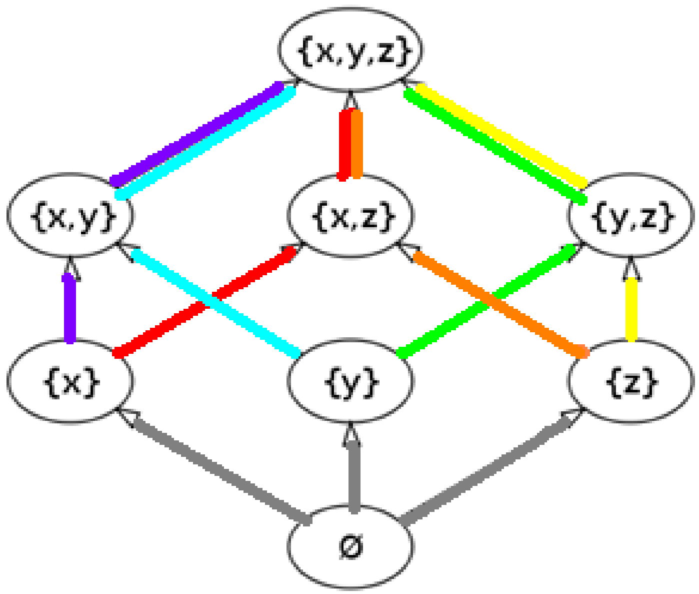
A **chain** (or totally ordered set) is a poset in which any two elements are comparable. Look like: $x_1 \leq x_2 \leq x_3 \leq x_4$

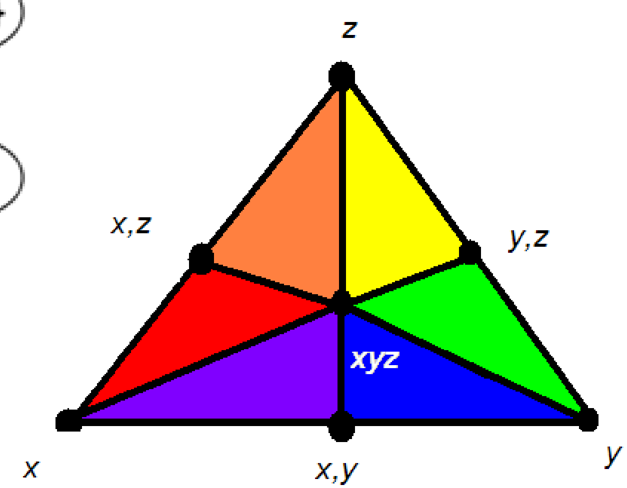
Order Complexes

There is a poset P naturally associated to every simplicial complex: the elements of the poset are faces, and the ordering is by inclusion.

Conversely, define Δ(P) to be the simplicial complex on the elements of P whose faces are all the chains of P - 0. Δ(P) is called the **order complex** of P.







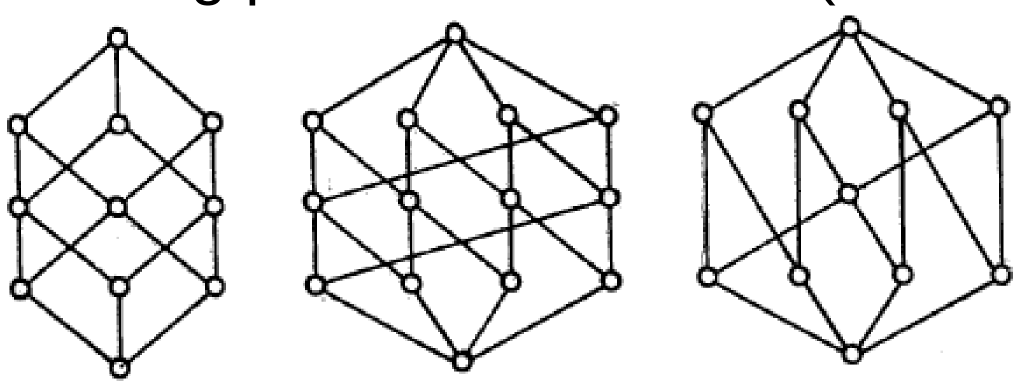
Lattices

A **lattice** is a poset where every pair of elements has a **least upper bound** (LUB) and a **greatest lower bound** (GLB)

A LUB, or **join** of x and y, written $z = x \vee y$, is a $z \in P$ such that $x \leq z$, $y \leq z$, and if $x \leq w$, $y \leq w$, then $z \leq w$.
$x \wedge y$ is the **meet** of *x* and *y*

All finite lattices have a 0 and a 1, elements that are universal upper or lower bounds.

Which of the following posets are lattices? (exercise in Stanley)



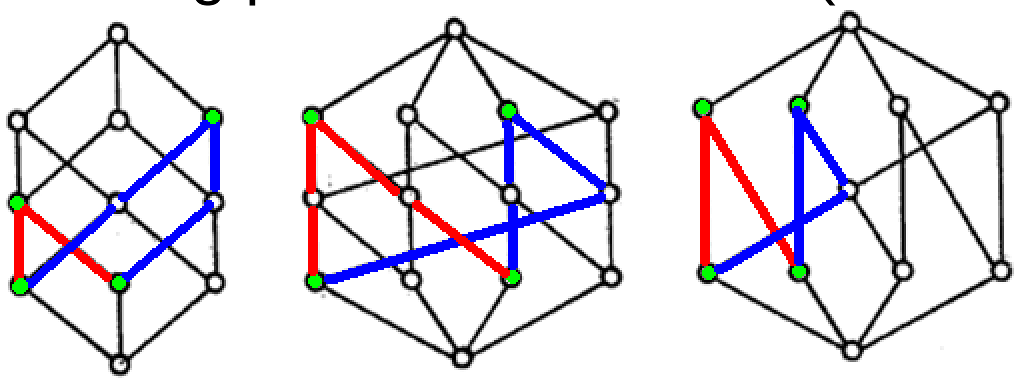
Lattices

A **lattice** is a poset where every pair of elements has a **least upper bound** (LUB) and a **greatest lower bound** (GLB)

A LUB, or **join** of x and y, written $z = x \vee y$, is a $z \in P$ such that $x \leq z$, $y \leq z$, and if $x \leq w$, $y \leq w$, then $z \leq w$.
$x \wedge y$ is the **meet** of *x* and *y*

All finite lattices have a 0 and a 1, elements that are universal upper or lower bounds.

Which of the following posets are lattices? (exercise in Stanley)



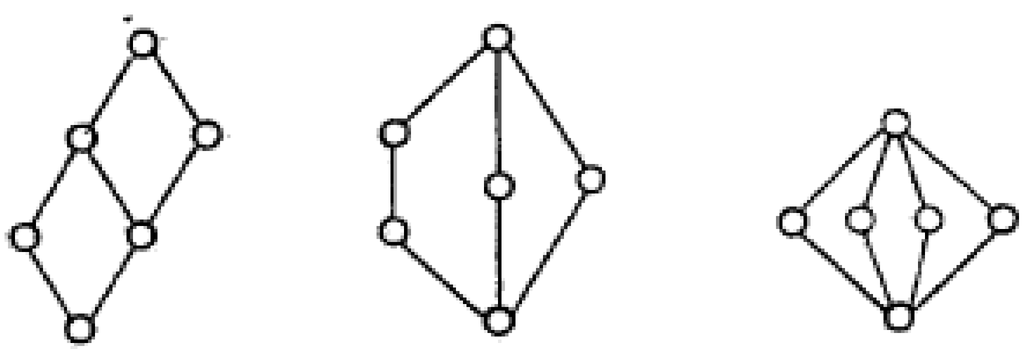
NONE

Examples of Lattices

The natural numbers form an infinite lattice with no “1”:
$a \leq b$ if $a \mid b$
$a \vee b$ is the least common multiple
$a \wedge b$ is the greatest common divisor

The lattice of subgroups of a group:
$H \leq K$ if H is contained in K
$H \vee K = \langle H, K \rangle$, subgroup generated by H and K
$H \wedge K = H \cap K$, check subgroups are closed under intersection

More Examples: (Stanley)



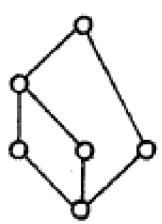
Types of Lattices

A poset/lattice is **graded** if every maximal chain has the same length n. Then there is a rank function ρ: P $\rightarrow \{1\ldots n\}$ such that $\rho(y) = x+1$ if y covers x and $\rho(x) = 0$ for minimal elements x.

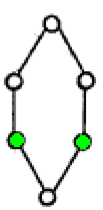
An **atom** is an element x of a lattice that covers 0 $\Leftrightarrow$ $\rho(x)=1$

A finite graded lattice is **semimodular** if
$\rho(x) + \rho(y) \geq \rho(x \wedge y) + \rho(x \vee y)$

Not graded:



Graded, not semimodular:



A finite graded lattice is **geometric** if it is semimodular and **atomic**, meaning every element is a join of atoms.

Matroids

A **matroid** is a pair (E, I), where E is a set, and I is a set of subsets of E. It is sometimes denoted M(E)

The elements in E are called **edges**

If a set of edges is in I, it is called **independent**.

The set I must obey certain axioms to be a matroid:

1) $\varnothing \in I$

2) If $A \in I$, and $B \subseteq A$, then $B \in I$

3) If $A, B \in I$ and $|B| < |A|$ then $\exists\, x \in A$ such that $(B \cup \{x\}) \in I$

Examples of Matroids

E := $\{v_{1,} \cdots, v_k\}$ are the columns of a matrix,
I := linearly independent subsets of E (over a given field)

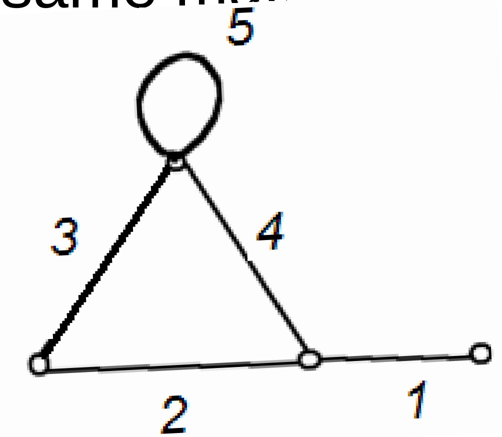
Any graph yields a matroid: the edges of the graph are the elements (edges) of the matroid

Independent sets are subsets of spanning trees, so a set is independent if and only if it contains no circuits.

Here are two representations of the same matroid:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

1 2 3 4 5



The Lattice of Flats of a Matroid

All matroids have a **rank** function that satisfies
$r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y) \;\; \forall\, X, Y \subseteq E$

A **closed set** or **flat** of a matroid is a set $X \subseteq E$ such that $\forall$
$e \in E$, $r(X) < r(X \cup \{e\})$.

Then the following is a lattice:
All flats of the matroid M(E)
$X \wedge Y = X \cap Y$
$X \vee Y = \text{closure}(X \cup Y)$

-This lattice is finite and semimodular
-Every element must be a join of atoms

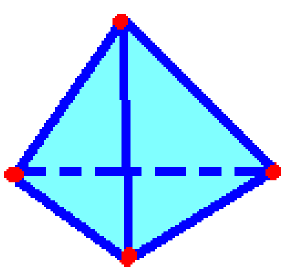
Fact: A lattice is geometric if and only if it is the lattice of flats of a matroid

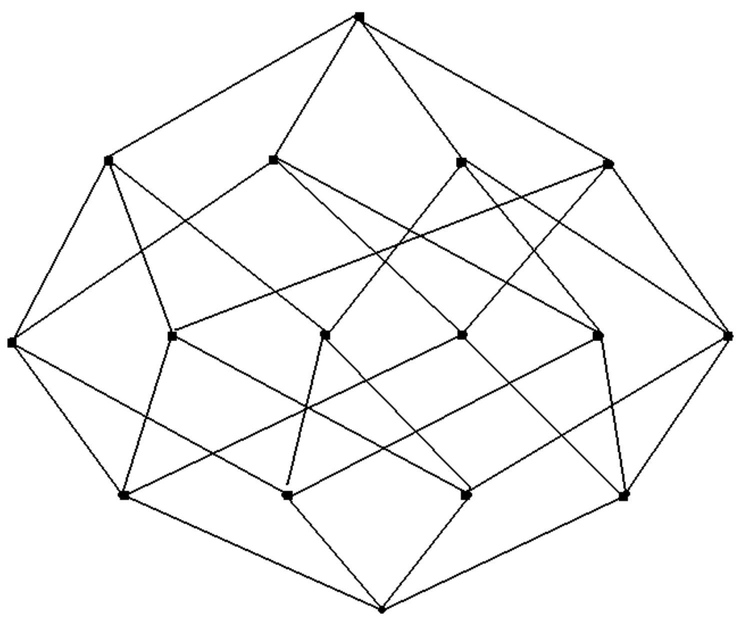
The Mobius Function of a Lattice

The Mobius function is the unique function $\mu: L \times L \rightarrow Z$ s.t.

1) If x and y are incomparable then $\mu(x,y) = 0$
2) $\mu(x, x) = 1$
3) If $x \leq y$ then $\sum_{x \leq z \leq y} (x,z) = 0$

This recursive definition is enough to compute values of μ on any lattice

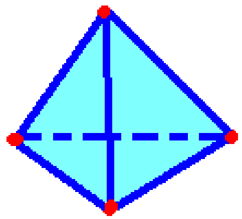




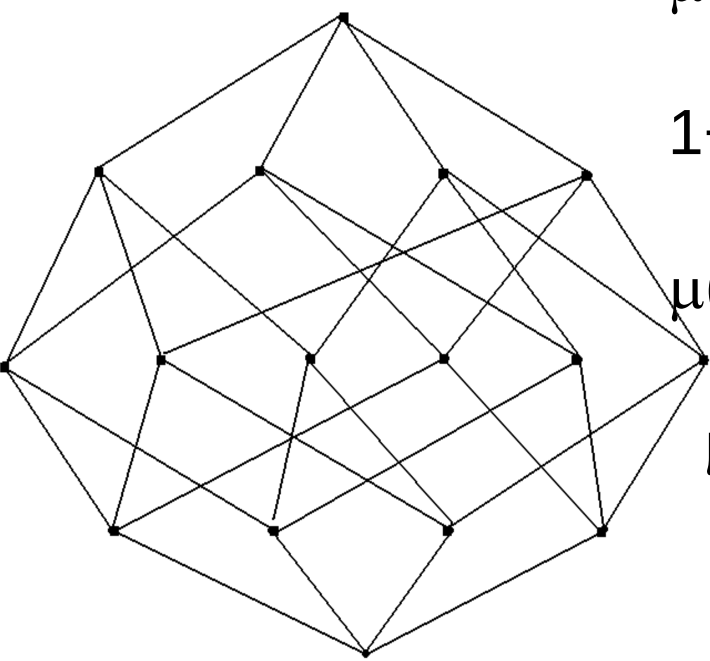
The Mobius Function of a Lattice

The Mobius function is the unique function μ: L $\times$ L $\rightarrow$ Z s.t.

1) If x and y are incomparable then μ(x,y) = 0
2) μ(x, x) = 1
3) If x $\leq$ y then $\sum_{x \leq z \leq y}$ (x,z) = 0



Use $\sum_{0 \leq z \leq y}$ μ(0,z) = 0 to compute values of μ from the bottom up



μ(0, 1)+1-4+6-4=0; μ(0, 1)= 1

1+3(-1)+3(1)+ μ(0, y)=0; μ(0, y)= -1

μ(0, 0)+μ(0, z_1)+μ(0, z_2)+ μ(0,y)=0; 1

μ(0, 0)+ μ(0, y)= 0; μ(0, y)= -1

μ(0,0)=1

The Mobius Function of a Lattice

Mobius functions of geometric lattices always alternate sign

Let L be a geometric lattice and $\bar{L} = L - \{0,1\}$
Then $\chi(\Delta(\bar{L})) = \mu(L)$

Use $\sum_{0 \leq z \leq y} \mu(0,z) = 0$ to compute values of μ from the bottom up

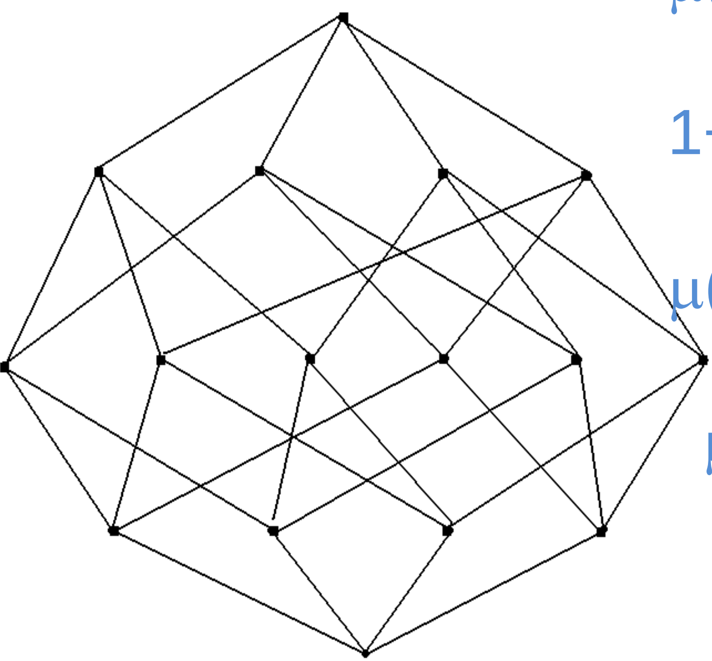
$\mu(0, 1)+1-4+6-4=0$; $\mu(0, 1)= -1$

$1+3(-1)+3(1)+ \mu(0, y)=0$; $\mu(0, y)= -1$

$\mu(0, 0)+\mu(0, z_1)+\mu(0, z_2)+ \mu(0,y)=0$; 1

$\mu(0, 0)+ \mu(0, y)= 0$; $\mu(0, y)= -1$

$\mu(0,0)=1$



The Mobius function on a Matroid

We define μ(M) to be μ(∅, E) on the lattice of flats of the M

If the empty set is not a flat (i.e. M contains a loop), then μ(M) = 0

If the empty set is a flat, then we can define the **characteristic polynomial** of M as:

$$\chi(M;t) = \sum_{A \in L_M} \mu(\emptyset, A) t^{r(M)-r(A)}$$

And another familiar friend appears:

$$\chi(M;t) = (-1)^{r(M)} T(M; 1-t, 0).$$

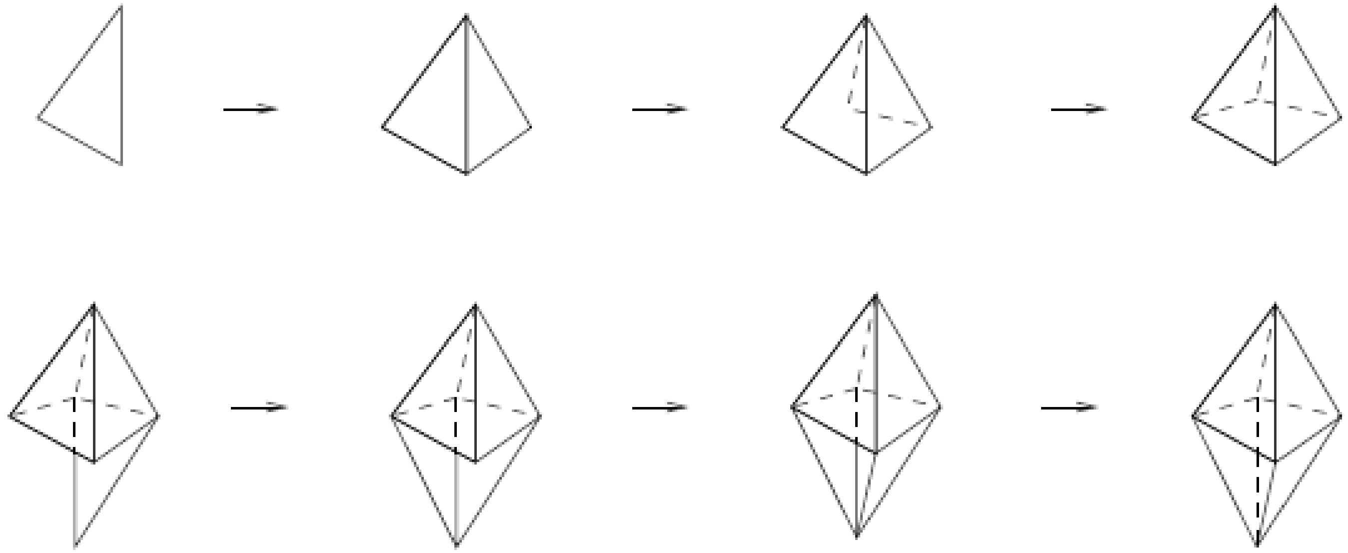
The Tutte Polynomial strikes again!

Shellability

A **linear ordering** of the facets is just a list: F_1, F_2, … F_k

A **shelling** is such an order with a special condition:
Pick a first facet. Now, each new facet added to the list must meet the old complex at a nonempty union of maximal proper faces.

A complex is **shellable** if it is pure and admits a shelling



Shellability

A **linear ordering** of the facets is just a list: $F_1, F_2, \ldots F_k$

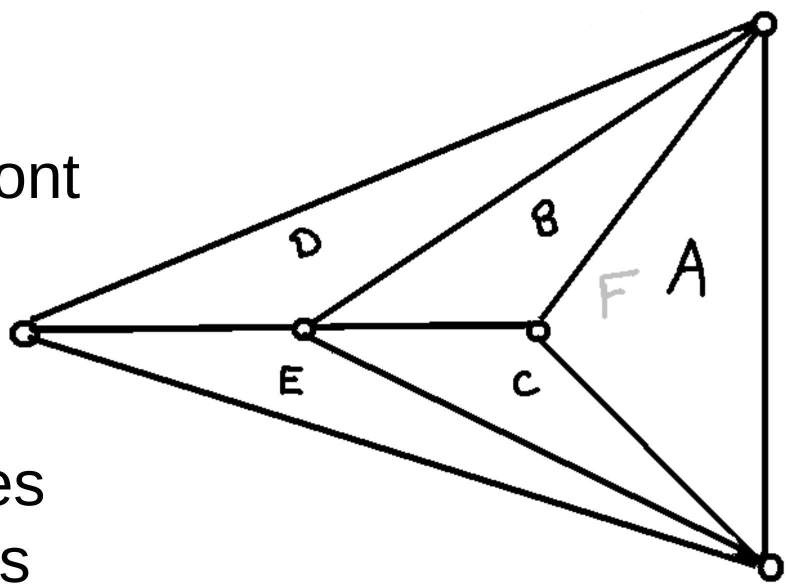
A **shelling** is such an order with a special condition:
Pick a first facet. Now, each new facet added to the list must meet the old complex at a nonempty union of maximal proper faces.

Bjorner example:

Facets are ABCDEF Where F is in front

Then ABCDEF is a shelling

C and E do not introduce new vertices
F does not even introduce new edges



Shellability

A **linear ordering** of the facets is just a list: F_1, F_2, … F_k

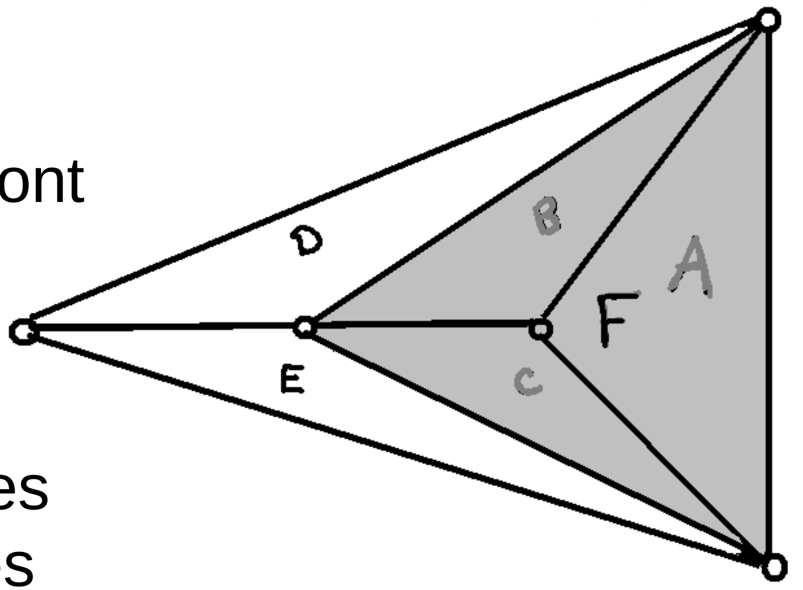
A **shelling** is such an order with a special condition:
Pick a first facet. Now, each new facet added to the list must meet the old complex at a nonempty union of maximal proper faces.

Bjorner example:

Facets are ABCDEF Where F is in front

Then ABCDEF is a shelling

C and E do not introduce new vertices
F does not even introduce new edges



Shellability

A **linear ordering** of the facets is just a list: $F_1, F_2, \ldots F_k$

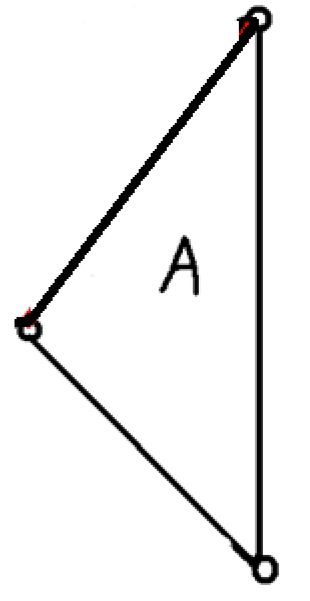
A **shelling** is such an order with a special condition:
Pick a first facet. Now, each new facet added to the list must meet the old complex at a nonempty union of maximal faces.

Bjorner example:

Facets are ABCDEF Where F is in front

Then ABCDEF is a shelling

C and E do not introduce new vertices
F does not even introduce new edges



Shellability

A **linear ordering** of the facets is just a list: $F_1, F_2, \ldots F_k$

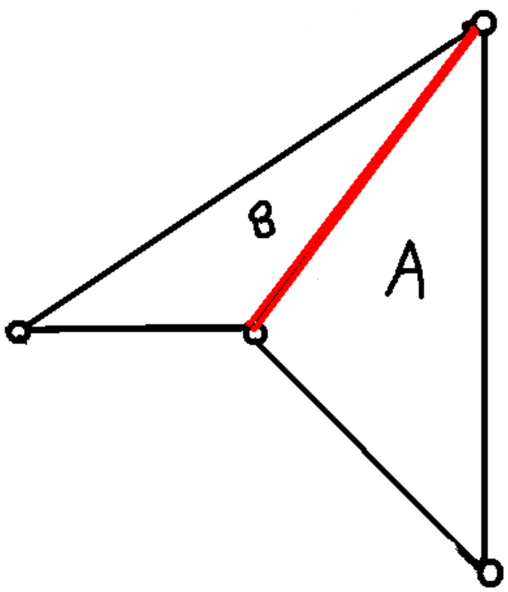
A **shelling** is such an order with a special condition:
Pick a first facet. Now, each new facet added to the list must meet the old complex at a nonempty union of maximal faces.

Bjorner example:

Facets are ABCDEF Where F is in front

Then ABCDEF is a shelling

C and E do not introduce new vertices
F does not even introduce new edges



Shellability

A **linear ordering** of the facets is just a list: $F_1, F_2, \ldots F_k$

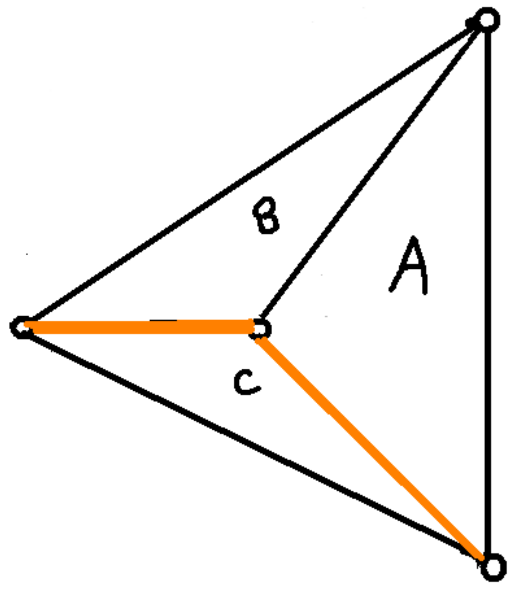
A **shelling** is such an order with a special condition:
Pick a first facet. Now, each new facet added to the list must meet the old complex at a nonempty union of maximal faces.

Bjorner example:

Facets are ABCDEF Where F is in front

Then ABCDEF is a shelling

C and E do not introduce new vertices
F does not even introduce new edges



Shellability

A **linear ordering** of the facets is just a list: $F_1, F_2, \dots F_k$

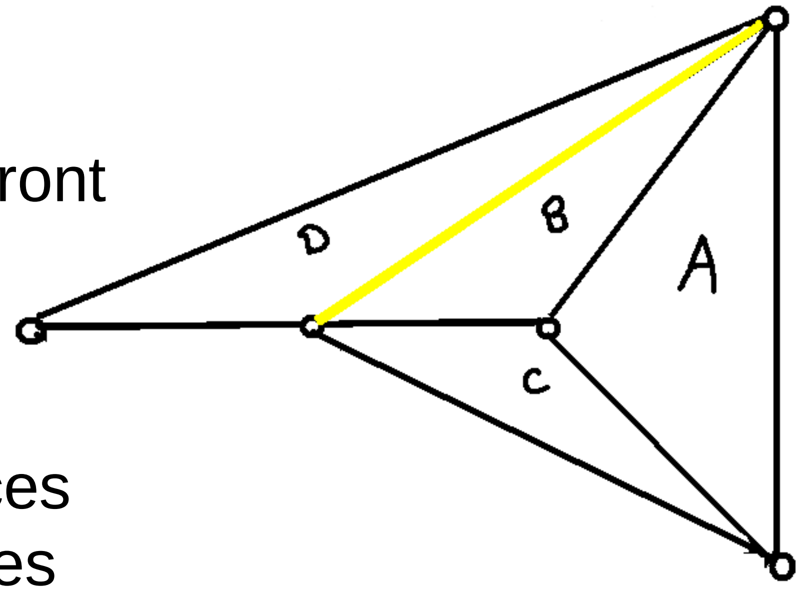
A **shelling** is such an order with a special condition:
Pick a first facet. Now, each new facet added to the list must meet the old complex at a nonempty union of maximal faces.

Bjorner example:

Facets are ABCDEF Where F is in front

Then ABCDEF is a shelling

C and E do not introduce new vertices
F does not even introduce new edges



Shellability

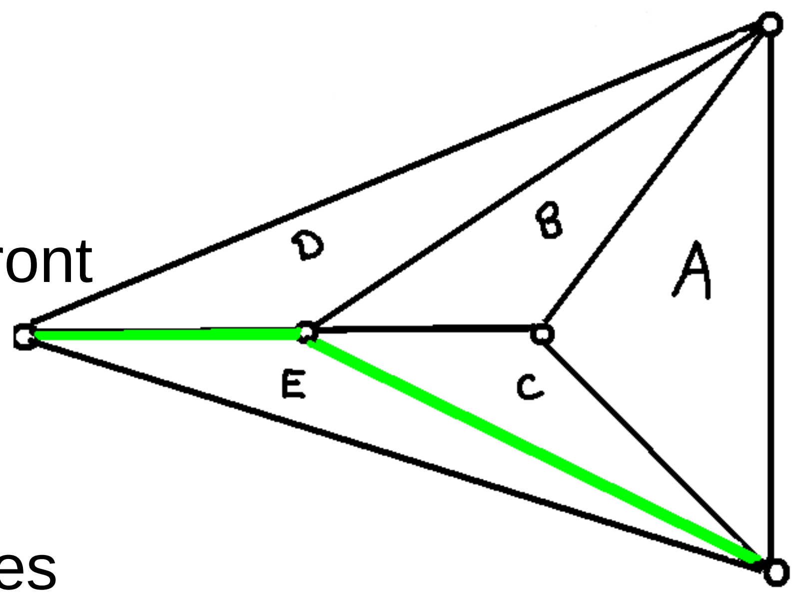
A **linear ordering** of the facets is just a list: $F_1, F_2, \dots F_k$

A **shelling** is such an order with a special condition:
Pick a first facet. Now, each new facet added to the list must meet the old complex at a nonempty union of maximal proper faces.

Bjorner example:

Facets are ABCDEF Where F is in front

Then ABCDEF is a shelling



C and E do not introduce new vertices
E does not even introduce new edges

Shellability

A **linear ordering** of the facets is just a list: $F_1, F_2, \ldots F_k$

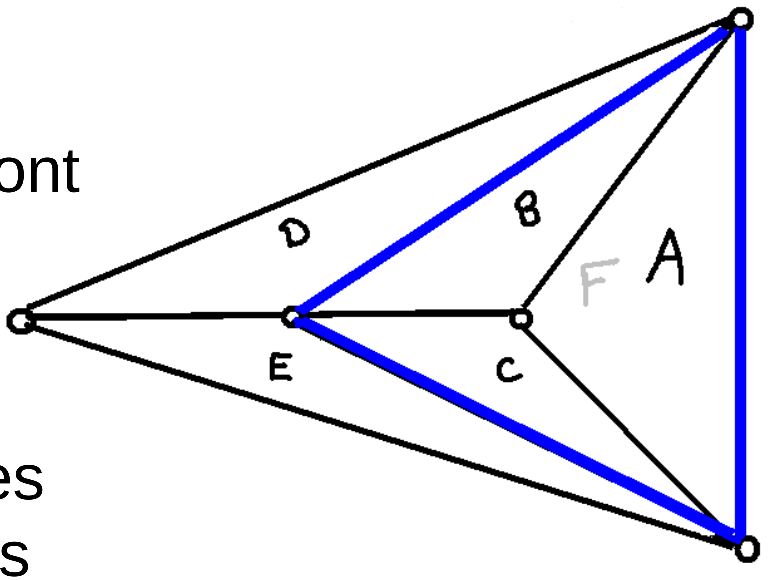
A **shelling** is such an order with a special condition:
Pick a first facet. Now, each new facet added to the list must meet the old complex at a nonempty union of maximal faces.

Bjorner example:

Facets are ABCDEF Where F is in front

Then ABCDEF is a shelling

C and E do not introduce new vertices
F does not even introduce new edges

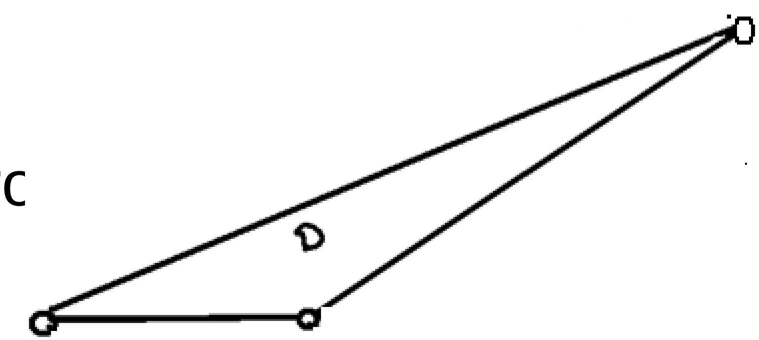


Shellability

A **linear ordering** of the facets is just a list: $F_1, F_2, \ldots F_k$

A **shelling** is such an order with a special condition:
Pick a first facet. Now, each new facet added to the list must meet the old complex at a nonempty union of maximal faces.

Bjorner example:



Facets are ABCDEF where F is in frc

DECABF is not a shelling

Exercise: ABEFDC is not a shelling

Shellability

A **linear ordering** of the facets is just a list: $F_1, F_2, \ldots F_k$

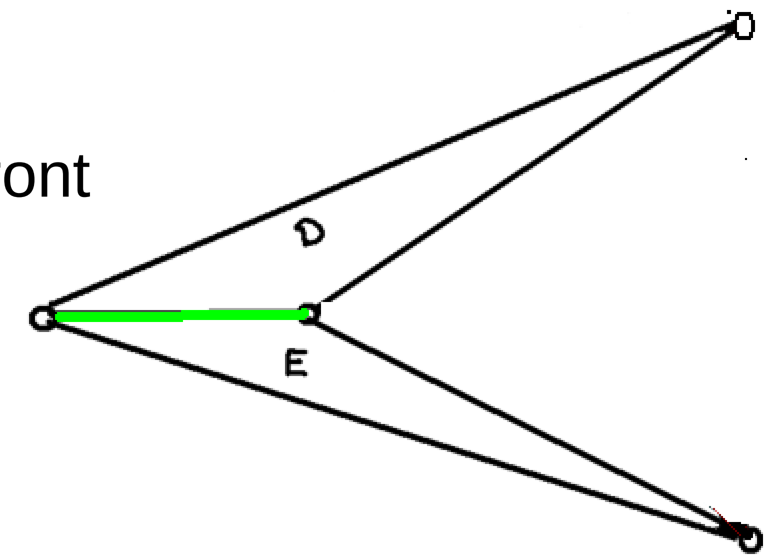
A **shelling** is such an order with a special condition:
Pick a first facet. Now, each new facet added to the list must meet the old complex at a nonempty union of maximal faces.

Bjorner example:

Facets are ABCDEF where F is in front

DECABF is not a shelling

Exercise: ABEFDC is not a shelling

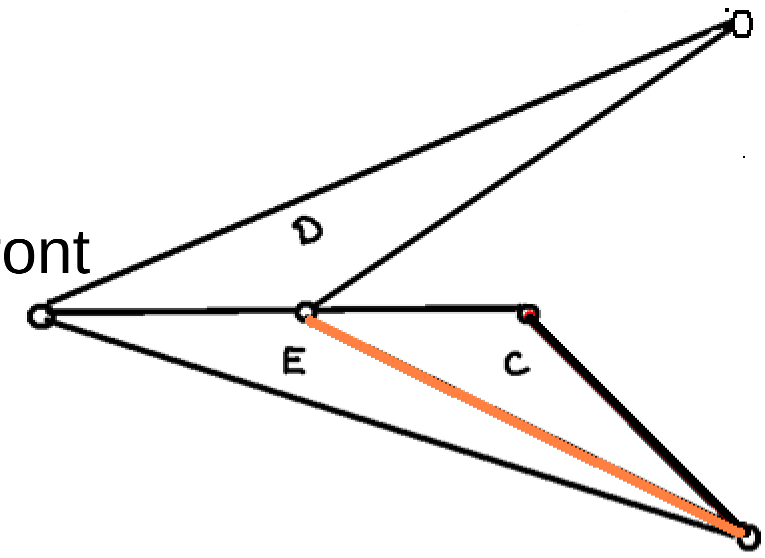


Shellability

A **linear ordering** of the facets is just a list: $F_1, F_2, \ldots F_k$

A **shelling** is such an order with a special condition:
Pick a first facet. Now, each new facet added to the list must meet the old complex at a nonempty union of maximal proper faces.

Bjorner example:



Facets are ABCDEF where F is in front

DECABF is not a shelling

Exercise: ABEFDC is not a shelling

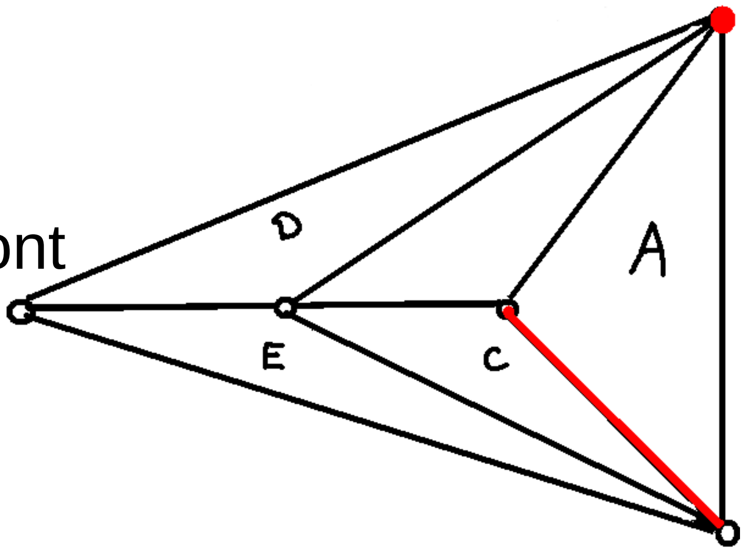
Shellability

A **linear ordering** of the facets is just a list: $F_1, F_2, \ldots F_k$

A **shelling** is such an order with a special condition:
Pick a first facet. Now, each new facet added to the list must meet the old complex at a nonempty union of maximal proper faces.

Bjorner example:

Facets are ABCDEF where F is in front



DEC<u>A</u>BF is not a shelling

Exercise: AB<u>E</u>FDC is not a shelling

Shellability

A **linear ordering** of the facets is just a list: $F_1, F_2, \ldots F_k$

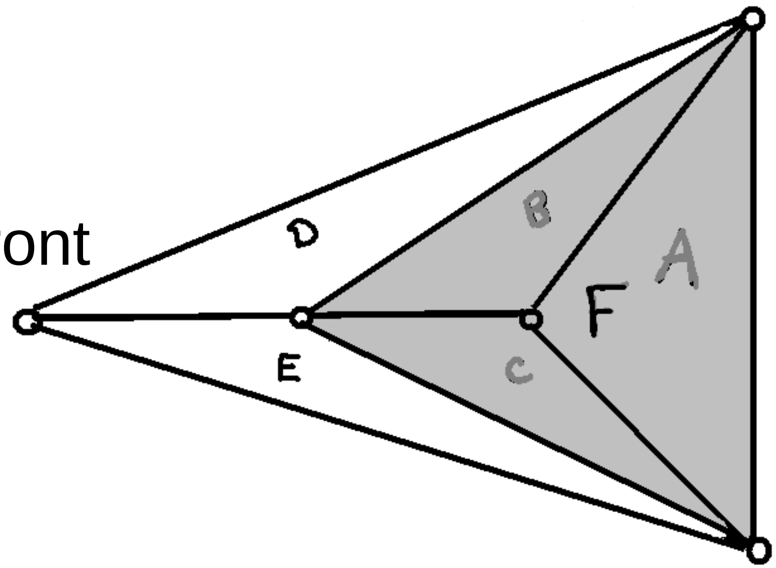
A **shelling** is such an order with a special condition:
Pick a first facet. Now, each new facet added to the list must meet the old complex at a nonempty union of maximal proper faces.

Bjorner example:

Facets are ABCDEF where F is in front

DECABF is not a shelling

Exercise: ABEFDC is not a shelling



More Shelling Facts

A graph (1-dimensional complex) is shellable if and only if it is connnected.

Any ordering of the facets of a simplex is a shelling

Every polytope (convex hull of points) is shellable, but order matters. In fact, not every partial shelling can be completed.

It's worth asking what sort of structure each new piece of the shelling is attached to. In the previous example, sometimes we glued new facets to a single edge, two edges, even 3 edges forming a triangle (in the case of the front face F).

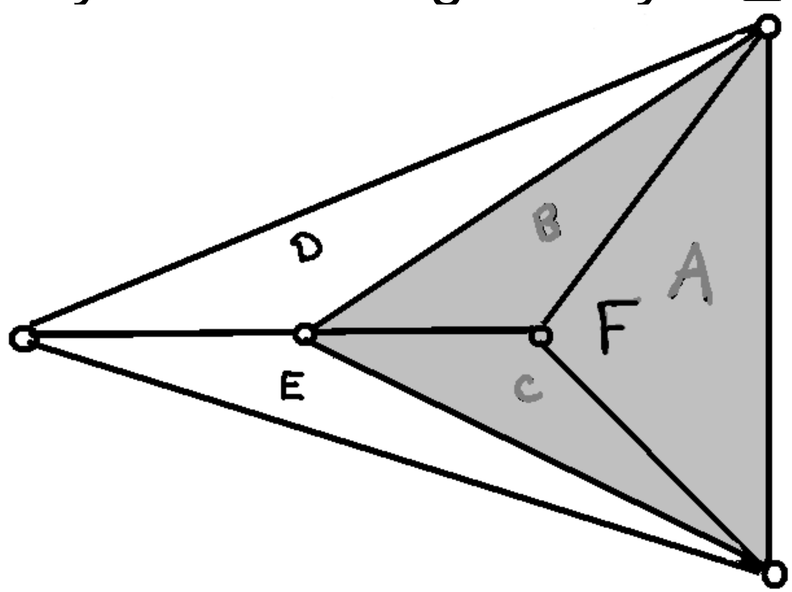
The Shelling Polynomial

Let the restriction of a facet $R(F_i) = \{x \in F_i : F_i \{ x \in \Delta_{i\text{-}1}\}$

$R(F_i) = \varnothing$ if and only if $i = 1$

$R(F_i) = F_i$ only if nothing new was added in lower dimension

The shelling polynomial h is given by $h_\Delta(x) = \sum_{i=1}^{k} x^{|F_i - R(F_i)|}$



h_Δ (x) =

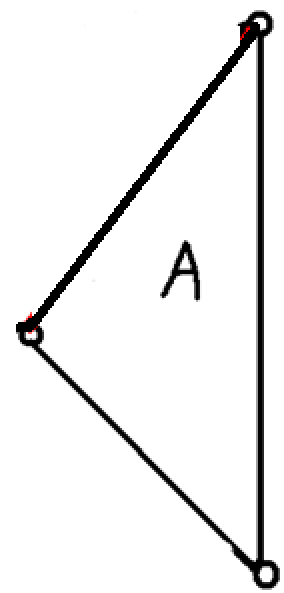
The Shelling Polynomial

Let the restriction of a facet $R(F_i) = \{x \in F_i : F_i \{ x \in \Delta_{i\text{-}1}\}$

$R(F_i) = \varnothing$ if and only if $i = 1$

$R(F_i) = F_i$ only if nothing new was added in lower dimension

The shelling polynomial h is given by $h_\Delta(x) = \sum_{i=1}^{k} x^{|F_i - R(F_i)|}$



$h_\Delta (x) = x^3$

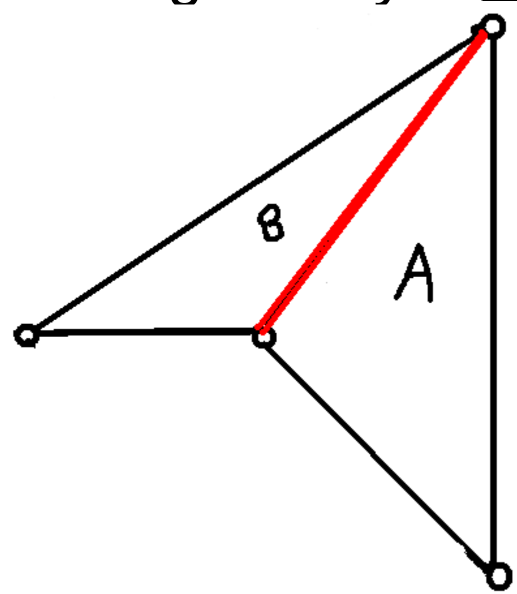
The Shelling Polynomial

Let the restriction of a facet $R(F_i) = \{x \in F_i : F_i \{ x \in \Delta_{i-1}\}$

$R(F_i) = \varnothing$ if and only if $i = 1$

$R(F_i) = F_i$ only if nothing new was added in lower dimension

The shelling polynomial h is given by $h_\Delta(x) = \sum_{i=1}^{k} x^{|F_i - R(F_i)|}$



$h_\Delta(x) = x^3 + x^2$

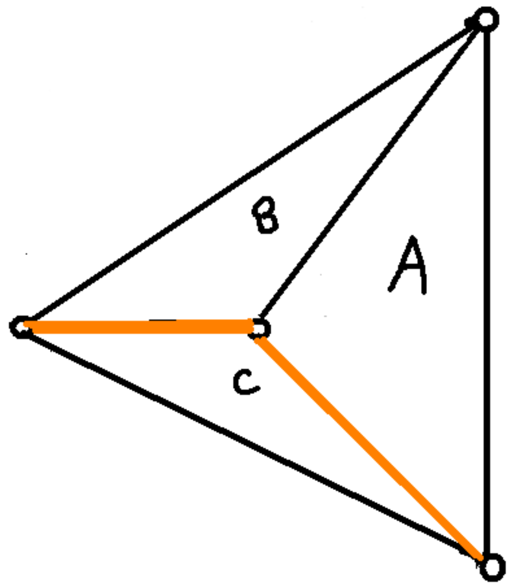
The Shelling Polynomial

Let the restriction of a facet $R(F_i) = \{x \in F_i : F_i \{ x \in \Delta_{i\text{-}1}\}$

$R(F_i) = \varnothing$ if and only if $i = 1$

$R(F_i) = F_i$ only if nothing new was added in lower dimension

The shelling polynomial h is given by $h_\Delta(x) = \sum_{i=1}^{k} x^{|F_i - R(F_i)|}$



$h_\Delta(x) = x^3 + x^2 + x$

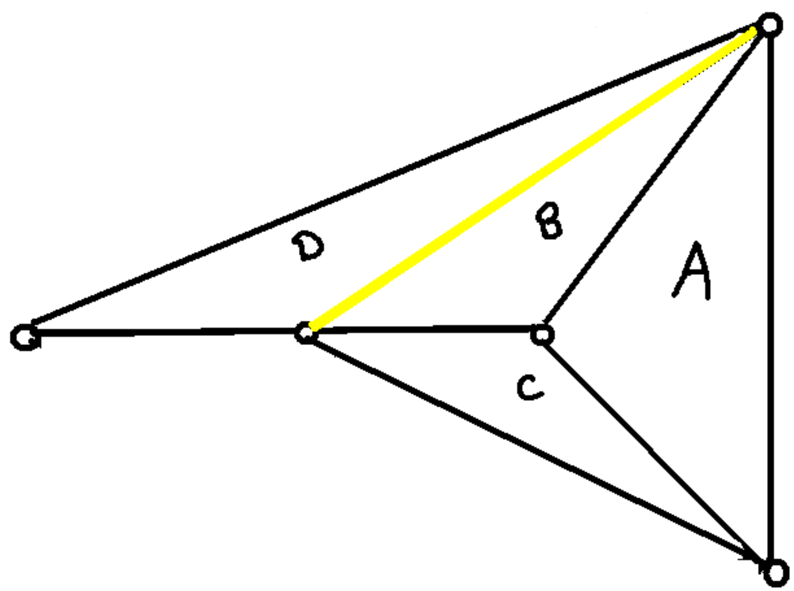
The Shelling Polynomial

Let the restriction of a facet $R(F_i) = \{x \in F_i : F_i \{ x \in \Delta_{i-1}\}$

$R(F_i) = \varnothing$ if and only if $i = 1$

$R(F_i) = F_i$ only if nothing new was added in lower dimension

The shelling polynomial h is given by $h_\Delta(x) = \sum_{i=1}^{k} x^{|F_i - R(F_i)|}$



$h_\Delta(x) = x^3 + x^2 + x + x^2$

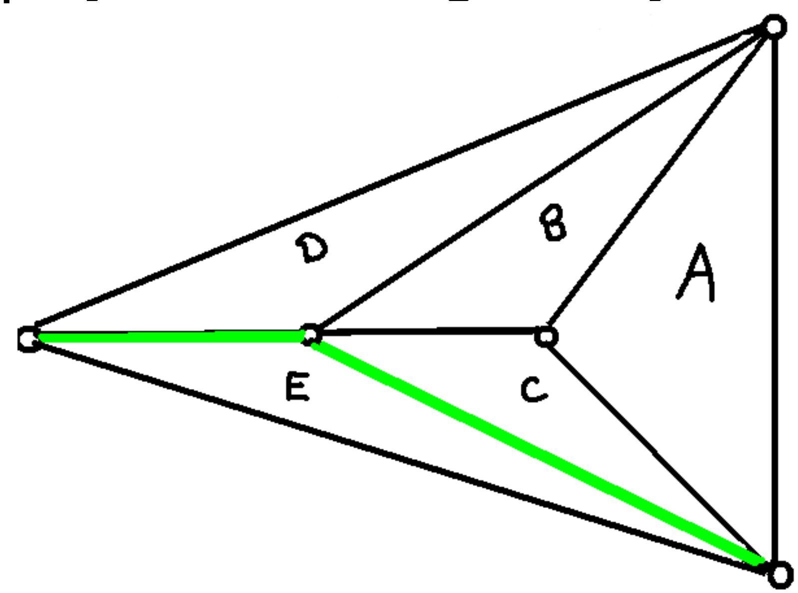
The Shelling Polynomial

Let the restriction of a facet $R(F_i) = \{x \in F_i : F_i \{ x \in \Delta_{i-1}\}$

$R(F_i) = \emptyset$ if and only if $i = 1$

$R(F_i) = F_i$ only if nothing new was added in lower dimension

The shelling polynomial h is given by $h_\Delta(x) = \sum_{i=1}^{k} x^{|F_i - R(F_i)|}$



$h_\Delta(x) = x^3 + x^2 + x + x^2 + x$

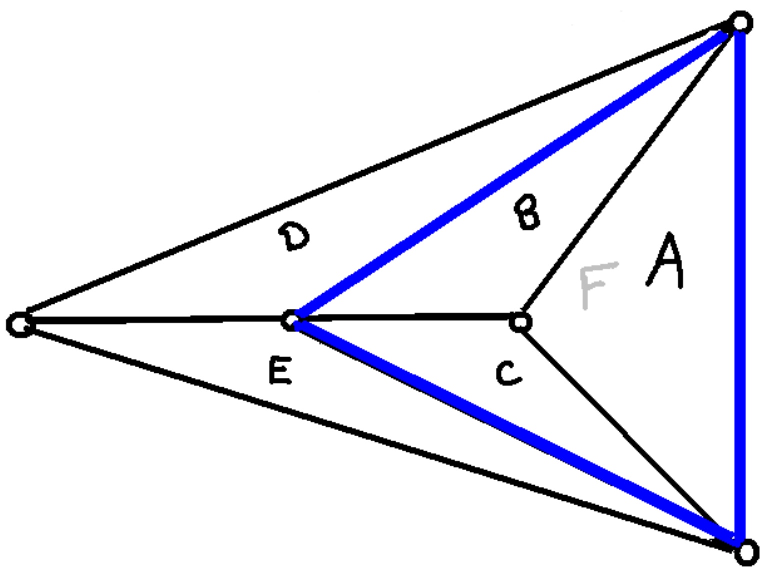
The Shelling Polynomial

Let the restriction of a facet $R(F_i) = \{x \in F_i : F_i \{ x \in \Delta_{i\text{-}1}\}$

$R(F_i) = \varnothing$ if and only if $i = 1$

$R(F_i) = F_i$ only if nothing new was added in lower dimension

The shelling polynomial h is given by $h_\Delta(x) = \sum_{i=1}^{k} x^{|F_i - R(F_i)|}$



$h_\Delta (x) = x^3 + x^2 + x + x^2 + x + 1 = x^3 + 2x^2 + 2x + 1$

The Shelling Polynomial

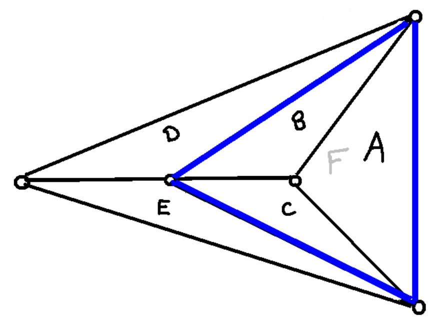
Let the restriction of a facet $R(F_i) = \{x \in F_i : F_i \{ x \in \Delta_{i-1}\}$

$$h_\Delta(x) = \sum_{i=1}^{k} x^{|F_i - R(F_i)|}$$

$h_\Delta(x) = x^3 + x^2 + x + x^2 + x + 1 = x^3 + 2x^2 + 2x + 1$

Theorem: If Δ is a shellable complex, then $f_\Delta(x) = h_\Delta(x+1)$

Here, $f_\Delta(x) = x^3 + 5x^2 + 9x + 6$



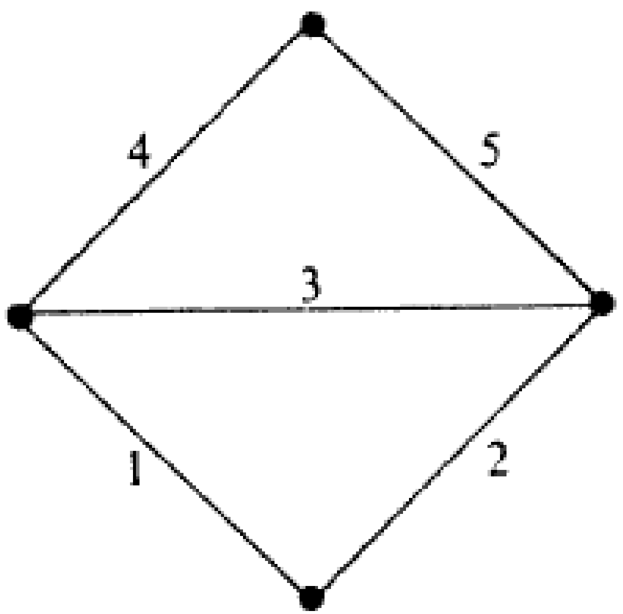
Corollary: $h_\Delta(0) = f_\Delta(-1) = (-1)^{r-1}\chi(\Delta)$

Note that $h_\Delta(0)$ is the constant term of $h_\Delta(x)$, so it represents all facets of the shelling where $F_i = R(F_i)$

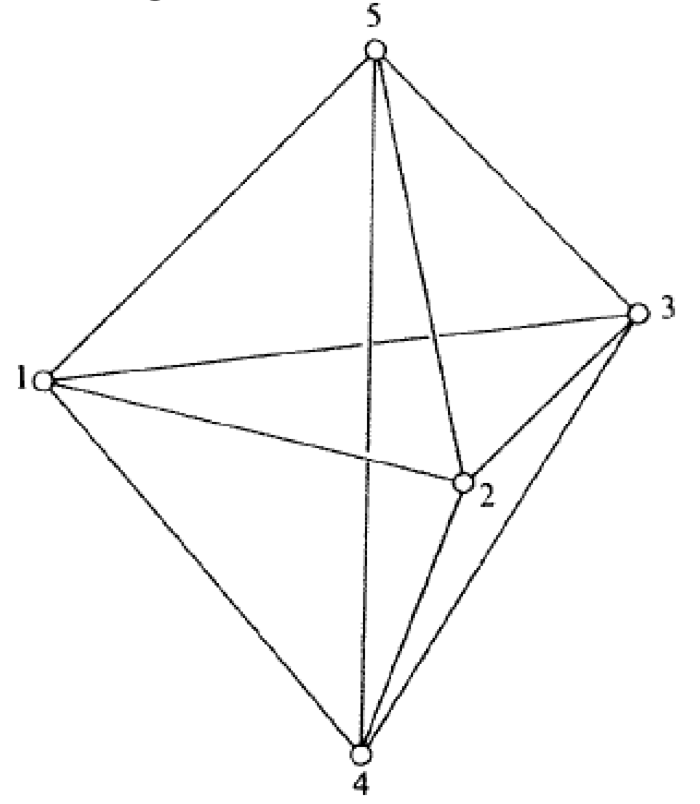
Matroid Complexes

If M is a matroid, then let IN(M) be the simplicial complex whose faces are the independent subsets of E

The matroid M represented graphically



IN(M) There are 2 interior triangles: 145 and 245



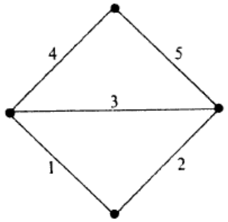
Matroid Complexes

If *M* is a matroid, then let *IN(M)* be the simplicial complex whose faces are the independent subsets of *E*

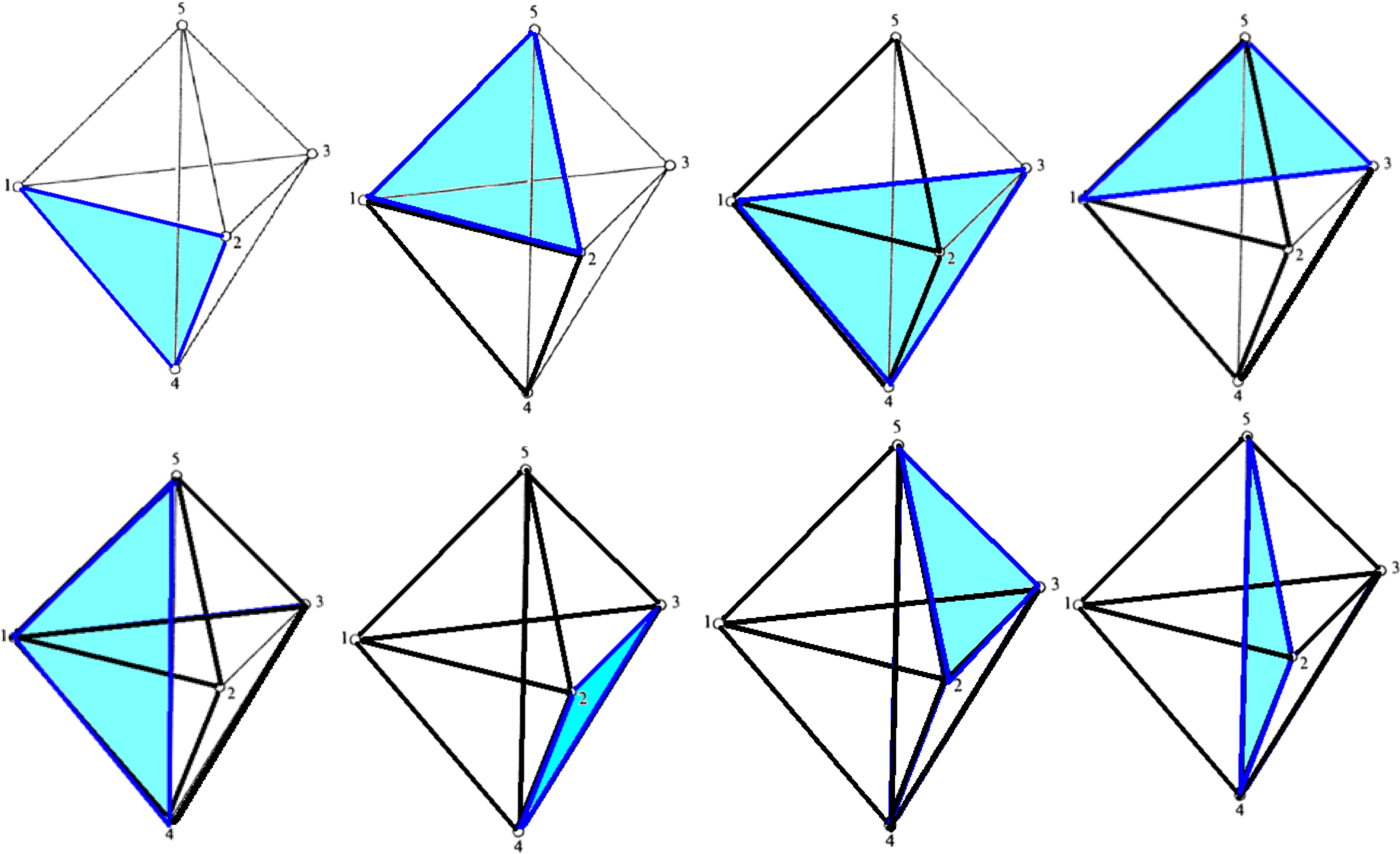
IN(M) is a pure complex, since every maximal independent set (basis) is the same size in a matroid

IN(M) is always shellable. Any linear ordering of the elements of M induces a lexicographic ordering on the bases, and this ordering of the facets of *IN(M)* is a shelling

In fact, a simplicial complex is a matroid complex if and only if every ordering of the vertices induces a lexicographic shelling.



The Lexicographic Shelling



The Shelling Polynomial and the Matroid

If *M* is a matroid, then let *IN(M)* be the simplicial complex whose faces are the independent subsets of *E*

Given an ordering of the elements, we can find R(B), the restriction of the facet representing the basis B, for any basis:

An element $e \in E\text{-}B$ is **externally active** if it is the least element of the circuit contained in $B \cup \{e\}$. Denote the number of externally active elements to a basis by ***e*(*B*)**

Dually, an element $e \in B$ is **internally active** if it is the least element of the bond (cycle in the dual matroid) $E\text{-}B \cup \{e\}$. Denote the number of internally active elements by ***i*(*B*).**

R(*B*)= *IP*(*B*), all the internally passive elements. These are the elements in the basis that are not "too small"

Shelling Polynomials Hit Matroids

Recall that R(*B*)= *IP*(*B*), $$h_\Delta(x) = \sum_{i=1}^{k} x^{|F_i - R(F_i)|}$$

and all our facets are bases now.

$$h_\Delta(x) = \sum_B x^{|B - \mathcal{R}(B)|} = \sum_B x^{|IA(B)|} = \sum_B x^{i(B)}$$

i(B)= internal activity
e(B)= external activity

Dually, $h_{\Delta^*}(y) = \sum_B y^{e(B)}$ where Δ^* is IN(M*)

Then $$T_M(x, y) = \sum_B x^{i(B)} y^{e(B)}$$

References

Bjorner, *The Homology and Shellability of Matroids and Geometric Lattices*

Swartz, *From Polytopes to Enumeration*

Stanley, *Enumerative Combinatorics*

Oxley, *Matroid Theory*

Ziegler, *Lectures on Polytopes*